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2006 J. Phys. A: Math. Gen. 39 5847

(http://iopscience.iop.org/0305-4470/39/20/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 5847–5856 doi:10.1088/0305-4470/39/20/014

Method for detecting concurrence without the structural physical approximation by local operations and classical communication

Yan-Kui Bai[1], **Shu-Shen Li**[1,2], **Hou-Zhi Zheng**[1,2] **and Ahmad Abliz**[3,4]

[1] State Key Laboratory for Superlattices and Microstructures, Institute of Semiconductors, Chinese Academy of Sciences, PO Box 912, Beijing 100083, People's Republic of China
[2] CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
[3] Nanoscale Physics and Devices Laboratory, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, People's Republic of China
[4] School of Mathematics, Physics and Information Sciences, Shinjiang Normal University, Urumchi 830054, People's Republic of China

E-mail: ykbai@semi.ac.cn

Received 29 December 2005, in final form 3 January 2006
Published 3 May 2006
Online at stacks.iop.org/JPhysA/39/5847

Abstract
We present a modified method for detecting the concurrence in an arbitrary two-qubit quantum state ρ_{AB} with local operations and classical communication. In this method, it is not necessary for the two observers to prepare the quantum state $\widetilde{\rho_{AB}}$ by the structural physical approximation. Their main task is to measure four specific functions via two local quantum networks. With these functions they can determine the concurrence and then the entanglement of formation.

PACS numbers: 03.67.Mn, 03.67.Lx, 03.67.Hk

1. Introduction

Characterizing the entanglement of a given quantum state is a fundamental problem in quantum information processing. When dealing with an unknown quantum state, we can resort to the quantum state tomography [1, 2], which provides the information about the density matrix. With the reconstructed quantum state, the entanglement property can be calculated in terms of some known measures and criteria. However, although it is simple, the use of quantum state tomography is not the most efficient method for the detection and measurement of entanglement. In recent years, several papers [3–10] have been devoted to the direct computation of the entanglement in an unknown quantum state without prior state reconstruction. Fewer parameters are required in these direct methods.

In the early methods [3–7], the observer is required to perform the partial transposition map or the transposition map on the unknown quantum state. Due to the non-physical property of these maps, the observer can only perform them with the aid of structural physical approximation (SPA) [11] which can change a non-physical map into a physical one. But the practical implementation of the SPA is difficult, which hinders the feasibility of these direct schemes. Recently, Carteret showed that it is not the only way to bring the quantum state under the maps when the observer wants to measure the effect of these maps on a quantum state [8]. She presented some networks for detecting [8] and measuring [10] entangled states without performing these maps. Carteret's methods are more feasible in practical application.

In quantum communication, the sender (Alice) and receiver (Bob) share a composite system, and they can perform only local operations and classical communication (LOCC). It is important in this scenario to detect and measure the entanglement of an unknown quantum state. For example, it has been shown that entanglement is a precondition for secure quantum key distribution [13]. An LOCC method for checking the Peres separability criterion [14, 15] without performing the SPA was presented in [9]. The method is more feasible than the early LOCC schemes [5, 6] in that it does not need SPA. In order to make better use of the entanglement, Alice and Bob need to consider further how much entanglement is there in their composite system. The entanglement of formation E_f is a well-defined measure of entanglement [16]. In particular, for a two-qubit mixed state, it has an analytical formula [12],

$$E_f(\rho_{AB}) = h\left(\frac{1+\sqrt{1-C(\rho_{AB})^2}}{2}\right), \tag{1}$$

where $h(x)$ is the Shannon function and C is the concurrence, which is defined as

$$C(\rho_{AB}) = \max[\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0], \tag{2}$$

in which the four monotonically decreasing real numbers λ_is are the eigenvalues of the matrix $\rho_{AB}\widetilde{\rho_{AB}}$, and here $\widetilde{\rho_{AB}} = (\sigma_y \otimes \sigma_y)\rho_{AB}^T(\sigma_y \otimes \sigma_y)$. As long as the concurrence is known, Alice and Bob can determine the entanglement of formation E_f. In [7], an LOCC method was presented for detecting the concurrence, which is an extension of Horodecki's method [4]. In the LOCC method, Alice and Bob can obtain the functions [4] $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ for $k = 1, 2, 3, 4$ by measuring seven parameters via two local networks (see figure 1 in [7]). With these functions, they can determine the concurrence and the entanglement of formation. However, the practical implementation of the above method is difficult. Because Alice and Bob need to prepare the quantum state $(\rho_{AB} \otimes \widetilde{\rho_{AB}})^{\otimes k}$, which requires the two observers to perform jointly the SPAs of two partial transposition maps.

In this paper, based on the global method [10] presented by Carteret, we present a modified LOCC method for detecting the concurrence. This method is more feasible than the previous LOCC method [7] because the SPA is not necessary. Moreover, we reduce the number of required parameters from seven to four by analysing the output state of the local networks. Our paper is organized as follows. In section 2, we present the LOCC method for detecting the concurrence without performing the SPA. Then we discuss our method in section 3. Finally, in section 4, we give some conclusions.

2. Detecting the concurrence without the SPA by LOCC

To see how Alice and Bob detect the concurrence without performing the SPA, we first recall the global method. In [10], Carteret presented some networks for measuring the functions $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ without preparing the quantum states $(\rho_{AB} \otimes \widetilde{\rho_{AB}})^{\otimes k}$. Her networks are inspired by a modified Mach–Zehnder interferometer network (see figure 1 in [17],

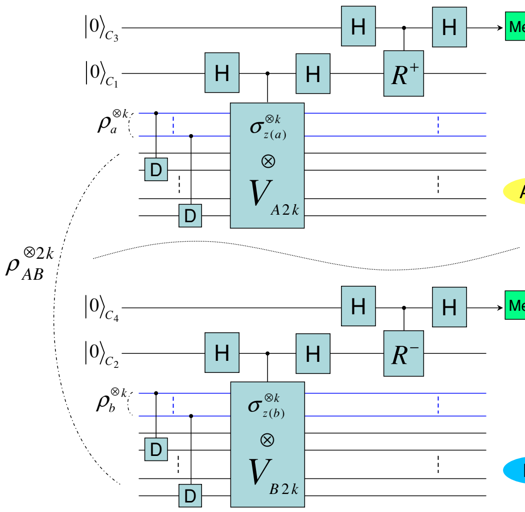

Figure 1. A general network for remotely detecting the concurrence. Alice and Bob can get the function $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ by measuring the control qubits c_3 and c_4.

cf [18, 19]) in which a controlled-U operation is inserted between two Hadamard gates. When one measures the control qubit in the computational basis, the modification of the interference is given by [17]

$$\mathrm{Tr}(U\rho) = v\mathrm{e}^{\mathrm{i}\alpha}, \tag{3}$$

where v is the visibility and α is the phase shift. In Carteret's networks (see figures 4 and 6 in [10]), the controlled-U gate is chosen to be a series of controlled-*Swap* gates, controlled–controlled-*Swap* gates and controlled-σ_z gates (the swap operation S is defined as $S|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle$). With the help of some ancillary qubits, the observer can obtain the visibility [10]

$$v_k = \frac{1}{4^k}\,\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k], \tag{4}$$

in which the relation [10, 21]

$$\widetilde{\rho_{AB}} = \rho_{AB} - \rho_A \otimes I_B - I_A \otimes \rho_B + I_{AB} \tag{5}$$

is used.

In this paper, based on Carteret's global method [10], we present an LOCC method for detecting the concurrence. We assume that Alice and Bob share a number of unknown quantum states ρ_{AB}. They first divide their ensemble into four groups. Then, in the kth group, they subdivide the quantum states into sets of $2k$ elements [7]. In our LOCC method, the main task for the two observers is to measure the functions $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ for $k = 1, 2, 3, 4$.

A general network used to accomplish the task is shown in figure 1 in which we assume that Alice and Bob have chosen a set of quantum states $\rho_{AB}^{\otimes 2k}$ in the kth group. The network is composed of two local networks, one for Alice and one for Bob. The first part of Alice's local network is a modified interferometer circuit in which a controlled-$\sigma_{z(a)}^{\otimes k} \otimes V_{A2k}$ operation is inserted between two Hadamard gates. Before entering the interferometer circuit, the input state $\rho_A^{\otimes 2k} \otimes \rho_a^{\otimes k}$ is initialized by some controlled-depolarizing operations. This is different from the global method [10] in which the initialization procedure is unnecessary. The purpose of the initialization operation is to simplify the interior circuit of the interferometer, i.e., after the initialization operation, Alice can perform the controlled-*Swap* operations instead of the controlled–controlled-*Swap* operations used in the global networks (see figures 4 and 6 in [10]). The second part[4] of Alice's network is another interferometer circuit in which a controlled-R^+ gate is inserted. The local network of Bob is the same to that of Alice except for the controlled-R^- gate in the second interferometer circuit. In the following analysis, we will show that Alice and Bob can get the function $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ by measuring the control qubits c_3 and c_4.

Now we analyse the first part of the LOCC network. In this part, the input state is

$$\begin{aligned}\rho_{in}(k) &= \rho_{c_1c_2} \otimes \rho_{abAB}(k)\\ &= \rho_{c_1} \otimes \rho_{c_2} \otimes \rho_a^{\otimes k} \otimes \rho_b^{\otimes k} \otimes \rho_{AB}^{\otimes 2k},\end{aligned} \tag{6}$$

where $\rho_{c_1} = |0\rangle\langle 0|$, $\rho_{c_2} = |0\rangle\langle 0|$ are the quantum states of the control qubits, and $\rho_a = |\phi\rangle\langle\phi|$, $\rho_b = |\phi\rangle\langle\phi|$ (here $|\phi\rangle = (|0\rangle + \sqrt{2}|1\rangle)/\sqrt{3}$) are the quantum states of the ancillary qubits. Before entering the two interferometer circuits, the quantum state $\rho_{abAB}(k)$ is subjected to $2k$ controlled-depolarizing operations. For these controlled-depolarizing operations shown in figure 1, the control system is the ancillary qubits $\rho_a^{\otimes k} \otimes \rho_b^{\otimes k}$ and the target is all the quantum states ρ_{AB} in the even position. The depolarizing map is defined as [11]

$$\Lambda_D(\rho) = I/d. \tag{7}$$

That is, it turns any ρ into the maximal mixed state. The circuit implementation of the controlled-depolarizing operation $\Lambda_{C_a-D_A}$ can be realized by a controlled-swap gate in which the state of the target qubit is swapped with a maximal mixed state [20],

$$\begin{aligned}\Lambda_{C_a-D_A}(\rho_a \otimes \rho_A) &= \mathrm{Tr}_{A'}\left[C_a - S_{AA'}\left(\rho_a \otimes \rho_A \otimes \frac{I_{A'}}{2}\right) C_a - S_{AA'}^{\dagger}\right]\\ &= \mathrm{Tr}_{A'}\left[\begin{pmatrix} I_{AA'} & 0\\ 0 & S_{AA'}\end{pmatrix}_a \begin{pmatrix} \frac{1}{3}\rho_A \otimes \frac{I_{A'}}{2} & 0\\ 0 & \frac{2}{3}\rho_A \otimes \frac{I_{A'}}{2}\end{pmatrix}_a \begin{pmatrix} I_{AA'} & 0\\ 0 & S_{AA'}\end{pmatrix}_a^{\dagger}\right]\\ &= \mathrm{Tr}_{A'}\left[\begin{pmatrix} \frac{1}{3}\rho_A \otimes \frac{I_{A'}}{2} & 0\\ 0 & \frac{1}{3}I_A \otimes \rho_{A'}\end{pmatrix}_a\right]\\ &= \frac{1}{3}\begin{pmatrix}\rho_A & 0\\ 0 & I_A\end{pmatrix}_a.\end{aligned} \tag{8}$$

In the above deduction, we only consider the diagonal elements of the ancillary qubit a. This is because, based on the principle of implicit measurement [20], we can assume the ancillary qubit a has been measured in the computational basis (in the following analysis, we can see the non-diagonal elements have no contribution to the quantum networks). Similarly, we can implement the controlled-depolarizing operation $\Lambda_{C_a-D_A\otimes C_b-D_B}(\rho_{ab} \otimes \rho_{AB})$ by performing two controlled-*Swap* gates:

$$\begin{aligned}\Lambda_{C_a-D_A\otimes C_b-D_B}(\rho_{ab} \otimes \rho_{AB}) = \mathrm{Tr}_{A'B'}\Bigg[&(C_a - S_{AA'} \otimes C_b - S_{BB'})\left(\rho_{ab} \otimes \rho_{AB} \otimes \frac{I_{A'}}{2} \otimes \frac{I_{B'}}{2}\right)\\ &\times\left(C_a - S_{AA'}^{\dagger} \otimes C_b - S_{BB'}^{\dagger}\right)\Bigg].\end{aligned} \tag{9}$$

[4] In [9], we made use of this part to measure the function $\mathrm{Tr}[(\rho_{AB}^{T_B})^k]$.

Passing through the $2k$ controlled-depolarizing channels, the global input state $\rho_{in}(k)$ will be

$$\begin{aligned}\rho'_{in}(k) &= \rho_{c_1c_2} \otimes \rho'_{abAB}(k)\\ &= \rho_{c_1c_2} \otimes \left\{\rho_{AB} \otimes \left[\Lambda_{C_a-D_A\otimes C_b-D_B}(\rho_a \otimes \rho_b \otimes \rho_{AB})\right]\right\}^{\otimes k}\\ &= \rho_{c_1c_2} \otimes \left[\rho_{AB} \otimes \frac{1}{9}\begin{pmatrix}\rho_{AB} & 0 & 0 & 0\\ 0 & \rho_A\otimes I_B & 0 & 0\\ 0 & 0 & I_A\otimes\rho_B & 0\\ 0 & 0 & 0 & I_{AB}\end{pmatrix}_{ab}\right]^{\otimes k},\end{aligned} \tag{10}$$

where we use the relations

$$\begin{aligned}\Lambda_{I_A\otimes I_B}(\rho_{AB}) &= \mathrm{Tr}_{A'B'}\left[(I_{AA'}\otimes I_{BB'})\left(\rho_{AB}\otimes\frac{I_{A'}}{2}\otimes\frac{I_{B'}}{2}\right)(I_{AA'}\otimes I_{BB'})^\dagger\right] = \rho_{AB}\\ \Lambda_{I_A\otimes D_B}(\rho_{AB}) &= \mathrm{Tr}_{A'B'}\left[(I_{AA'}\otimes S_{BB'})\left(\rho_{AB}\otimes\frac{I_{A'}}{2}\otimes\frac{I_{B'}}{2}\right)(I_{AA'}\otimes S_{BB'})^\dagger\right] = \rho_A\otimes\frac{I_B}{2},\\ \Lambda_{D_A\otimes I_B}(\rho_{AB}) &= \mathrm{Tr}_{A'B'}\left[(S_{AA'}\otimes I_{BB'})\left(\rho_{AB}\otimes\frac{I_{A'}}{2}\otimes\frac{I_{B'}}{2}\right)(S_{AA'}\otimes I_{BB'})^\dagger\right] = \frac{I_A}{2}\otimes\rho_B\\ \Lambda_{D_A\otimes D_B}(\rho_{AB}) &= \mathrm{Tr}_{A'B'}\left[(S_{AA'}\otimes S_{BB'})\left(\rho_{AB}\otimes\frac{I_{A'}}{2}\otimes\frac{I_{B'}}{2}\right)(S_{AA'}\otimes S_{BB'})^\dagger\right] = \frac{I_A}{2}\otimes\frac{I_B}{2}.\end{aligned} \tag{11}$$

In equation (10), based on the output state of the operation $\Lambda_{C_a-D_A\otimes C_b-D_B}(\rho_{ab}\otimes\rho_{AB})$, we can deduce that the quantum state ρ_{AB} in even position will evolve into

$$\rho'_{AB} = \tfrac{1}{9}(\rho_{AB} + \rho_A\otimes I_B + I_A\otimes\rho_B + I_{AB}). \tag{12}$$

Although the quantum state ρ'_{AB} is different from $\widetilde{\rho_{AB}}$ in equation (5) which have two minus signs, Alice and Bob can measure the function of quantum state $\widetilde{\rho_{AB}}$ by performing two controlled-σ_z operations in the interior circuit of subsequent interferometers [10].

The quantum state $\rho'_{in}(k)$ will be input subsequently into the two local interferometer circuits in which the Hadamard gate and the controlled-U gate can be written as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad U_{C-U} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\otimes I + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\otimes U. \tag{13}$$

Passing through the two local circuits, The quantum state $\rho_{c_1c_2}$ will be

$$\rho'_{c_1c_2}(k) = \mathrm{Tr}_{abAB}\left[U_{h_{12}}U_{c-\sigma v}U_{h_{12}}\rho'_{in}(k)U^\dagger_{h_{12}}U^\dagger_{c-\sigma v}U^\dagger_{h_{12}}\right], \tag{14}$$

where $U_{h_{12}} = H_{c_1}\otimes H_{c_2}\otimes I_{ab}^{\otimes k}\otimes I_{AB}^{\otimes 2k}$ and $U_{c-\sigma v} = U_{C_{c_1}-[\sigma_{z(a)}^{\otimes k}\otimes V_{A2k}]}\otimes U_{C_{c_2}-[\sigma_{z(b)}^{\otimes k}\otimes V_{B2k}]}$, in which the unitary shift operator V_k is defined as [17] $V_k|\phi_1\rangle|\phi_2\rangle\cdots|\phi_k\rangle = |\phi_k\rangle|\phi_1\rangle\cdots|\phi_{k-1}\rangle$. After a similar deduction as equation (13) in [7], we can obtain

$$\rho'_{c_1c_2}(k) = \frac{1}{4}\begin{pmatrix}1+\mu_1^{(k)}+\mu_3^{(k)} & \mu_5^{(k)} & -\mu_5^{(k)} & \mu_4^{(k)}\\ -\mu_5^{(k)} & 1+\mu_2^{(k)}-\mu_3^{(k)} & -\mu_4^{(k)} & \mu_5^{(k)}\\ \mu_5^{(k)} & -\mu_4^{(k)} & 1-\mu_2^{(k)}-\mu_3^{(k)} & -\mu_5^{(k)}\\ \mu_4^{(k)} & -\mu_5^{(k)} & \mu_5^{(k)} & 1-\mu_1^{(k)}+\mu_3^{(k)}\end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}&\mu_1^{(k)} = \xi_1^{(k)}+\xi_2^{(k)}, \qquad \mu_2^{(k)} = \xi_1^{(k)}-\xi_2^{(k)}, \qquad \mu_3^{(k)} = \tfrac{1}{2}\xi_3^{(k)}+\tfrac{1}{4}\xi_4^{(k)}+\tfrac{1}{4}\xi_5^{(k)},\\ &\mu_4^{(k)} = \tfrac{1}{2}\xi_3^{(k)}-\tfrac{1}{4}\xi_4^{(k)}-\tfrac{1}{4}\xi_5^{(k)}, \qquad \mu_5^{(k)} = \tfrac{1}{4}\xi_4^{(k)}-\tfrac{1}{4}\xi_5^{(k)},\end{aligned} \tag{16}$$

in which

$$\begin{aligned}
\xi_1^{(k)} &= \mathrm{Tr}_{abAB}\{[(\sigma_{z(a)}^{\otimes k}\otimes V_{A2k})\otimes I_b^{\otimes k}\otimes I_B^{\otimes 2k}]\rho'_{abAB}(k)\}\\
\xi_2^{(k)} &= \mathrm{Tr}_{abAB}\{[I_a^{\otimes k}\otimes I_A^{\otimes 2k}\otimes(\sigma_{z(b)}^{\otimes k}\otimes V_{B2k})]\rho'_{abAB}(k)\}\\
\xi_3^{(k)} &= \mathrm{Tr}_{abAB}\{[(\sigma_{z(a)}^{\otimes k}\otimes V_{A2k})\otimes(\sigma_{z(b)}^{\otimes k}\otimes V_{B2k})]\rho'_{abAB}(k)\}\\
\xi_4^{(k)} &= \mathrm{Tr}_{abAB}\{[(\sigma_{z(a)}^{\otimes k}\otimes V_{A2k})^{\dagger}\otimes(\sigma_{z(b)}^{\otimes k}\otimes V_{B2k})]\rho'_{abAB}(k)\}\\
\xi_5^{(k)} &= \mathrm{Tr}_{abAB}\{[(\sigma_{z(a)}^{\otimes k}\otimes V_{A2k})\otimes(\sigma_{z(b)}^{\otimes k}\otimes V_{B2k})^{\dagger}]\rho'_{abAB}(k)\}.
\end{aligned}\tag{17}$$

Before analysing the second part of the LOCC network, we first analyse the functions in equation (17). Having considered the expression of the quantum state $\rho'_{abAB}(k)$ in equation (10), we have

$$\begin{aligned}
\xi_3^{(k)} &= \mathrm{Tr}_{abAB}\left\{V_{A2k}\otimes V_{B2k}\left[\rho_{AB}\otimes\frac{1}{9}\begin{pmatrix}\rho_{AB} & 0 & 0 & 0\\ 0 & -\rho_A\otimes I_B & 0 & 0\\ 0 & 0 & -I_A\otimes\rho_B & 0\\ 0 & 0 & 0 & I_{AB}\end{pmatrix}_{ab}\right]^{\otimes k}\right\}\\
&= \mathrm{Tr}_{AB}\left\{V_{A2k}\otimes V_{B2k}\left[\rho_{AB}\otimes\frac{1}{9}\widetilde{\rho_{AB}}\right]^{\otimes k}\right\}\\
&= \frac{1}{9^k}\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k],
\end{aligned}\tag{18}$$

in which we have made use of equation (5) and the property [4] $\mathrm{Tr}(V_k A_1\otimes A_2\otimes\cdots\otimes A_k) = \mathrm{Tr}(A_1A_2\cdots A_k)$. (Here, we can see the non-diagonal element of the ancillary qubits a and b has no contribution to the function $\xi_3^{(k)}$.) In a similar way, we get

$$\xi_1^{(k)} = \frac{1}{3^k}\mathrm{Tr}[(\rho_A^2-\rho_A)^k], \qquad \xi_2^{(k)} = \frac{1}{3^k}\mathrm{Tr}[(\rho_B^2-\rho_B)^k]. \tag{19}$$

For the function $\xi_4^{(k)}$, after tracing out the ancillary qubits a and b, we obtain

$$\xi_4^{(k)} = \frac{1}{9^k}\mathrm{Tr}[V_{A2k}^{\dagger}\otimes V_{B2k}(\rho_{AB}\otimes\widetilde{\rho_{AB}})^{\otimes k}], \tag{20}$$

in which the effect of the quantum operation $V_{A2k}^{\dagger}\otimes V_{B2k}$ [9] can be expressed as

$$\begin{aligned}
&\mathrm{Tr}[V_{A2k}^{\dagger}\otimes V_{B2k}(\rho_{AB}\otimes\widetilde{\rho_{AB}})^{\otimes k}]\\
&\quad= \mathrm{Tr}\{\Sigma\rho_{i_1j_1}^{m_1n_1}\tilde{\rho}_{i_2j_2}^{m_2n_2}\rho_{i_3j_3}^{m_3n_3}\cdots\tilde{\rho}_{i_{2k}j_{2k}}^{m_{2k}n_{2k}}|i_1j_{2k}\rangle\langle m_{2k}n_1|\\
&\qquad\otimes|i_2j_1\rangle\langle m_1n_2|\otimes|i_3j_2\rangle\langle m_2n_3|\otimes\cdots\otimes|i_{2k}j_{2k-1}\rangle\langle m_{2k-1}n_{2k}|\}\\
&\quad= \sum\rho_{i_1j_1}^{i_2j_{2k}}\tilde{\rho}_{i_2j_2}^{i_3j_1}\rho_{i_3j_3}^{i_4j_2}\cdots\tilde{\rho}_{i_{2k}j_{2k}}^{i_1j_{2k-1}}.
\end{aligned}\tag{21}$$

Combining the above relation with the definition of the partial transposition $(\rho_{ij}^{mn})^{T_B} = \rho_{in}^{mj}$, we can rewrite the function $\xi_4^{(k)}$ as

$$\xi_4^{(k)} = \frac{1}{9^k}\mathrm{Tr}[(\rho_{AB}^{T_B}\tilde{\rho}_{AB}^{T_B})^k]. \tag{22}$$

For the function $\xi_5^{(k)}$, after a similar analysis, we get

$$\xi_5^{(k)} = \frac{1}{9^k}\mathrm{Tr}[(\rho_{AB}^{T_A}\tilde{\rho}_{AB}^{T_A})^k]. \tag{23}$$

In fact, the functions $\xi_4^{(k)}$ and $\xi_5^{(k)}$ are equivalent. This is because

$$\begin{aligned}\xi_5^{(k)} &= \frac{1}{9^k}\mathrm{Tr}\{[(\rho_{AB}^{T_B})^T(\widetilde{\rho}_{AB}^{T_B})^T]^k\}\\ &= \frac{1}{9^k}\mathrm{Tr}\{[(\widetilde{\rho}_{AB}^{T_B}\rho_{AB}^{T_B})^k]^T\}\\ &= \frac{1}{9^k}\mathrm{Tr}[(\rho_{AB}^{T_B}\widetilde{\rho}_{AB}^{T_B})^k] = \xi_4^{(k)}.\end{aligned} \tag{24}$$

Thus, in equation (15), the parameter $\mu_5^{(k)} = 0$, and the quantum state $\rho'_{c_1c_2}(k)$ can be rewritten as

$$\rho'_{c_1c_2}(k) = \frac{1}{4}\begin{pmatrix} 1+\mu_1^{(k)}+\mu_3^{(k)} & 0 & 0 & \mu_4^{(k)} \\ 0 & 1+\mu_2^{(k)}-\mu_3^{(k)} & -\mu_4^{(k)} & 0 \\ 0 & -\mu_4^{(k)} & 1-\mu_2^{(k)}-\mu_3^{(k)} & 0 \\ \mu_4^{(k)} & 0 & 0 & 1-\mu_1^{(k)}+\mu_3^{(k)} \end{pmatrix},$$

where

$$\begin{aligned}\mu_1^{(k)} &= \frac{1}{3^k}\{\mathrm{Tr}[(\rho_A^2-\rho_A)^k]+\mathrm{Tr}[(\rho_B^2-\rho_B)^k]\}\\ \mu_2^{(k)} &= \frac{1}{3^k}\{\mathrm{Tr}[(\rho_A^2-\rho_A)^k]-\mathrm{Tr}[(\rho_B^2-\rho_B)^k]\}\\ \mu_3^{(k)} &= \frac{1}{2\cdot 9^k}\{\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]+\mathrm{Tr}[(\rho_{AB}^{T_B}\widetilde{\rho}_{AB}^{T_B})^k]\}\\ \mu_4^{(k)} &= \frac{1}{2\cdot 9^k}\{\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]-\mathrm{Tr}[(\rho_{AB}^{T_B}\widetilde{\rho}_{AB}^{T_B})^k]\}.\end{aligned} \tag{25}$$

Now we analyse the second part of the LOCC network. As shown in figure 1, this part is composed of two local interferometer circuits, in which a controlled-R^+ gate and a controlled-R^- gate are inserted, respectively, where [9]

$$R^+ = \frac{1}{\sqrt{2}}(\sigma_z+\sigma_y) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & -1\end{pmatrix}, \qquad R^- = \frac{1}{\sqrt{2}}(\sigma_z-\sigma_y) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}. \tag{26}$$

(In [9], the two interferometer circuits were used to measure the function $\mathrm{Tr}[(\rho_{AB}^{T_B})^k]$. In this paper, we use them again). The input state for this part is

$$\rho''_{\mathrm{in}}(k) = \rho_{c_3c_4}\otimes\rho'_{c_1c_2}(k), \tag{27}$$

in which $\rho_{c_3c_4} = |00\rangle\langle 00|$ is the initial state of the control qubits c_3 and c_4. Passing through the two interferometer circuits, the quantum state $\rho_{c_3c_4}$ will transform into

$$\rho_{c_3c_4}^{\mathrm{out}}(k) = \mathrm{Tr}_{c_1c_2}[U_{h_{34}}U_rU_{h_{34}}\rho''_{\mathrm{in}}(k)U_{h_{34}}^\dagger U_r^\dagger U_{h_{34}}^\dagger], \tag{28}$$

where $U_{h_{34}} = H_{c_3}\otimes H_{c_4}\otimes I_{c_1c_2}$ and $U_r = U_{C_{c_3}-R_{c_1}^+}\otimes U_{C_{c_4}-R_{c_2}^-}$. After some deduction, we can get

$$\rho_{c_3c_4}^{\mathrm{out}}(k) = \frac{1}{4}\begin{pmatrix} 1+\tau_1^{(k)}+\eta^{(k)} & 0 & 0 & 0\\ 0 & 1+\tau_2^{(k)}-\eta^{(k)} & 0 & 0\\ 0 & 0 & 1-\tau_2^{(k)}-\eta^{(k)} & 0\\ 0 & 0 & 0 & 1-\tau_1^{(k)}+\eta^{(k)}\end{pmatrix}, \tag{29}$$

in which $\tau_1^{(k)} = \frac{1}{\sqrt{2}}\mu_1^{(k)}$, $\tau_2^{(k)} = \frac{1}{\sqrt{2}}\mu_2^{(k)}$ and $\eta^{(k)} = \frac{1}{2\cdot 9^k}\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$. If Alice and Bob measure the probabilities $P_{c_3c_4}(ij)$ of the quantum state $\rho_{c_3c_4}^{\mathrm{out}}(k)$ to be found in the state

$|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$, respectively, they can determine the function $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$. This is because

$$\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k] = 2\times 9^k \eta^{(k)} = 2\times 9^k\big[P_{c_3c_4}(00) - P_{c_3c_4}(01) - P_{c_3c_4}(10) + P_{c_3c_4}(11)\big]. \tag{30}$$

When Alice and Bob choose different input states $\rho_{\mathrm{in}}(k)$, they can obtain the functions $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ for $k = 1, 2, 3, 4$. With these functions, they can determine the concurrence and then the entanglement of formation. This concludes our general description of the modified LOCC method.

3. Discussion

In our LOCC method, the main task for Alice and Bob is to measure the functions $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$, for $k = 1, 2, 3, 4$. Of all these measurements, the case for $k = 1$ is special. On the one hand, Alice and Bob can get this function by measuring the control qubits c_1 and c_2, which means the second part of the LOCC network is not necessary. This is because, combining the Hermitian property of V_2 [11] with equations (20) and (22), we can obtain

$$\mathrm{Tr}[\rho_{AB}\widetilde{\rho_{AB}}] = \mathrm{Tr}\big[\rho_{AB}^{T_B}\tilde{\rho}_{AB}^{T_B}\big]. \tag{31}$$

Therefore, the parameter $\mu_4^{(1)} = 0$, and equation (25) can be rewritten as

$$\begin{aligned}\rho'_{c_1c_2}(1) = \tfrac{1}{12}\{&\big[1 + \mathrm{Tr}\big(\rho_A^2\big) + \mathrm{Tr}\big(\rho_B^2\big) + \tfrac{1}{3}\mathrm{Tr}(\rho_{AB}\widetilde{\rho_{AB}})\big]|00\rangle\langle 00|\\ &+\big[3 + \mathrm{Tr}\big(\rho_A^2\big) - \mathrm{Tr}\big(\rho_B^2\big) - \tfrac{1}{3}\mathrm{Tr}(\rho_{AB}\widetilde{\rho_{AB}})\big]|01\rangle\langle 01|\\ &+\big[3 - \mathrm{Tr}\big(\rho_A^2\big) + \mathrm{Tr}\big(\rho_B^2\big) - \tfrac{1}{3}\mathrm{Tr}(\rho_{AB}\widetilde{\rho_{AB}})\big]|10\rangle\langle 10|\\ &+\big[5 - \mathrm{Tr}\big(\rho_A^2\big) - \mathrm{Tr}\big(\rho_B^2\big) + \tfrac{1}{3}\mathrm{Tr}(\rho_{AB}\widetilde{\rho_{AB}})\big]|11\rangle\langle 11|\}.\end{aligned} \tag{32}$$

In this case, Alice and Bob can get the function $\mathrm{Tr}[\rho_{AB}\widetilde{\rho_{AB}}]$ by measuring the probabilities $P_{c_1c_2}(00)$, $P_{c_1c_2}(01)$, $P_{c_1c_2}(10)$ and $P_{c_1c_2}(11)$, respectively. On the other hand, Alice and Bob can omit the ancillary qubits a and b and the controlled-depolarizing operations shown in figure 1. From equation (5), they have [10]

$$\mathrm{Tr}[\rho_{AB}\widetilde{\rho_{AB}}] = \mathrm{Tr}\big[\rho_{AB}^2\big] - \mathrm{Tr}[\rho_{AB}(\rho_A\otimes I_B)] - \mathrm{Tr}[\rho_{AB}(I_A\otimes\rho_B)] + 1, \tag{33}$$

in which

$$\begin{aligned}\mathrm{Tr}[\rho_{AB}(\rho_A\otimes I_B)] &= \mathrm{Tr}\big[\Sigma\rho_{i_1j_1}^{m_1n_1}\rho_{i_2r_2}^{m_2r_2}\delta_{j_2}^{n_2}|i_1j_1\rangle\langle m_1n_1|i_2j_2\rangle\langle m_2n_2|\big]\\ &= \sum\rho_{i_1j_1}^{i_2j_1}\rho_{i_2j_2}^{i_1j_2} = \mathrm{Tr}\big[\rho_A^2\big],\\ \mathrm{Tr}[\rho_{AB}(I_A\otimes\rho_B)] &= \mathrm{Tr}\big[\Sigma\rho_{i_1j_1}^{m_1n_1}\delta_{i_2}^{m_2}\rho_{r_2j_2}^{r_2n_2}|i_1j_1\rangle\langle m_1n_1|i_2j_2\rangle\langle m_2n_2|\big]\\ &= \sum\rho_{i_1j_1}^{i_1j_2}\rho_{i_2j_2}^{i_2j_1} = \mathrm{Tr}\big[\rho_B^2\big].\end{aligned} \tag{34}$$

Therefore, the function $\mathrm{Tr}[\rho_{AB}\widetilde{\rho_{AB}}]$ is determined by $\mathrm{Tr}\big[\rho_{AB}^2\big]$, $\mathrm{Tr}\big[\rho_A^2\big]$ and $\mathrm{Tr}\big[\rho_B^2\big]$. When Alice and Bob omit the ancillary qubits and the controlled-depolarizing operation, the first part of our LOCC network will be the same to Alves's network (see figure 2 in [5]). In this case, the quantum state $\rho'_{c_1c_2}(1)$ will be [5]

$$\begin{aligned}\rho'_{c_1c_2}(1) = \tfrac{1}{4}\{&\big[1 + \mathrm{Tr}\big(\rho_A^2\big) + \mathrm{Tr}\big(\rho_B^2\big) + \mathrm{Tr}\big(\rho_{AB}^2\big)\big]|00\rangle\langle 00|\\ &+\big[1 + \mathrm{Tr}\big(\rho_A^2\big) - \mathrm{Tr}\big(\rho_B^2\big) - \mathrm{Tr}\big(\rho_{AB}^2\big)\big]|01\rangle\langle 01|\\ &+\big[1 - \mathrm{Tr}\big(\rho_A^2\big) + \mathrm{Tr}\big(\rho_B^2\big) - \mathrm{Tr}\big(\rho_{AB}^2\big)\big]|10\rangle\langle 10|\\ &+\big[1 - \mathrm{Tr}\big(\rho_A^2\big) - \mathrm{Tr}\big(\rho_B^2\big) + \mathrm{Tr}\big(\rho_{AB}^2\big)\big]|11\rangle\langle 11|\}.\end{aligned} \tag{35}$$

The function $\mathrm{Tr}\left[\rho_{AB}^2\right]$ can be obtained by measuring the probabilities $P_{c_1c_2}(ij)$ [5]. As a byproduct, the functions $\mathrm{Tr}\left[\rho_A^2\right]$ and $\mathrm{Tr}\left[\rho_B^2\right]$ can be obtained in the following way. In the measurement, Alice measures her qubit c_1 first, and then Bob measures his qubit c_2. Thus, Alice can obtain the function $\mathrm{Tr}\left[\rho_A^2\right]$ in terms of the relation $P_{c_1}(0) = \frac{1}{2} + \frac{1}{2}\mathrm{Tr}\left[\rho_A^2\right]$. Then Alice and Bob can deduce the function $\mathrm{Tr}\left[\rho_B^2\right]$ in terms of the relation $P_{c_1c_2}(00) = \frac{1}{4}\left[1 + \mathrm{Tr}\left(\rho_A^2\right) + \mathrm{Tr}\left(\rho_B^2\right) + \mathrm{Tr}\left(\rho_{AB}^2\right)\right]$.

In quantum measurement, it should be noted that the matrix elements of an unknown quantum state can be determined precisely only if an infinite ensemble of identically prepared quantum states is given [22]. But an infinite ensemble cannot be obtained in practice. Thus, the observer can only determine the matrix elements approximately with a finite ensemble. In our LOCC method, the functions $\mathrm{Tr}[(\rho_{AB}\widetilde{\rho_{AB}})^k]$ are determined by some diagonal elements (the probabilities $P(ij)$) of the quantum state $\rho'_{c_1c_2}(1)$ and $\rho^{\mathrm{out}}_{c_3c_4}(k)(k \geqslant 2)$. In order to get these diagonal elements with high fidelity, Alice and Bob need to run the LOCC network shown in figure 1 many times. Especially, for $k \geqslant 2$, Alice and Bob need to run the network even more times. This is because the function $\eta^{(k)}$ in equation (29) is a small quantity, which demands that Alice and Bob determine the probabilities $P_{c_3c_4}(ij)$ with a higher fidelity.

In figure 1, with the aid of the $2k$ controlled-depolarizing channels, the interior circuit of the modified interferometer is simplified. The controlled–controlled-*Swap* gates used in the global method [10] are replaced by the controlled-*Swap* gates. This simplification also results in a smaller parameter $\eta^{(k)}$ compared with the visibility v_k in equation (4). But, unlike the global method, the two observers do not need to measure the parameter $\eta^{(k)}$ directly. They can obtain $\eta^{(k)}$ by measuring the probabilities $P(ij)$ via a larger ensemble of identically prepared quantum states.

There is a potential problem in the practical application of our LOCC method. The difficulty comes from the effective implementation of the controlled quantum operations in experiment, especially the three-qubit controlled-*Swap* gate [23]. The solution of this problem relies on the quantum technology that is currently being developed.

4. Conclusion

In this paper, based on the global method [10] presented by Carteret, we present a modified LOCC method for detecting the concurrence. Unlike the previous LOCC method [7], our modified method does not require Alice and Bob to perform the SPA. Moreover, we reduce the number of parameters that need to be measured from 7 to 4 by reanalysing the output state of control qubits. These two improvements make the method more feasible in practice. Finally, with our modified method, we further validate Carteret's conclusion [8] in the LOCC scenario, i.e., when Alice and Bob want to measure the function of quantum state $\widetilde{\rho_{AB}}$, making the quantum state ρ_{AB} undergo the map $\Lambda(\rho_{AB}) = \widetilde{\rho_{AB}}$ is not the only way.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 60325416 and 60521001) and the Special Foundation for State Major Basic Research Program of China (Grant No. G2001CB309500).

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